JOURNAL OF APPROXIMATION THEORY **84**, 41–60 (1996)
Article No. 0004

Hermite Interpolation in the Roots of Unity

T. N. T. Goodman

Department of Mathematics and Computer Science, The University, Dundee DD1 4HN, Scotland

K. G. Ivanov*

Institute of Mathematics, Bulgarian Academy of Sciences, Sofia, Bulgaria

AND

A. Sharma

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

Communicated by András Kroó

Received February 22, 1994; accepted in revised form February 6, 1995

DEDICATED TO WALEED AL-SALAAM

We study the polynomial $H_{r,n}(f, z)$ which interpolates an analytic function f and its derivatives up to order $r-1$ at the nth roots of unity. In particular we relate the vanishing of the coefficients of the highest powers of z in the Hermite interpolant $H_{r,n}(f, z)$ with the vanishing at certain points of the Hermite interpolants of certain functions related to f.

1. Introduction

Several results of Walsh's theory of equiconvergence [9] (see also [1, 6]) show the close behaviour of $s_{n-1}(f, z)$, the Taylor polynomial of degree $n-1$ of a function f, and the Lagrange interpolant to f on the zeros of $z^n - \rho^n$, $L_{n-1,\rho}(f, z)$. For example, if f is analytic on $|z| \leqslant 1$, i.e., analytic on $|z| < 1 + \varepsilon$ for some $\varepsilon > 0$, then for any z, $\{L_{n-1,1}(f, z)\}_1^\infty$ and $\{s_{n-1}(f, z)\}_1^\infty$ either both convergence or both diverge. Moreover, if f lies in A_0, the class of functions analytic in $|z| < 1$ and continuous but not analytic on $|z| \leqslant 1$, then for any $0 < \rho < 1$, $\lim_{n\to\infty}(L_{n-1,\rho}(f, z) - s_{n-1}(f, z))$

* Supported by Grant MM-205/92 of the National Foundation for Scientific Investigations, Bulgaria.

0021-9045/96 $12.00

$=0$ for $|z|<1/\rho^2$. However, in [4] Ivanov and Saff showed that while $\{s_{n-1}(f, z)\}_1^\infty$ must diverge for f in A_0 and $|z|>1$, it is possible for any $|z|>1$ to find a function f in A_0 for which $\{L_{n-1,1}(f, z)\}_1^\infty$ is identically zero. This result is a corollary of the following theorem.

THEOREM A. *Let Λ be any subset of $\mathbb{N}$ and let $m \in \mathbb{N}$. The following are equivalent*:

(a) *There exists an $f \in A_0$ such that the first m coefficients $C(j, n)$, $j=n-1, ..., n-m$ of $L_{n-1}(z, f)$ are zeros for every $n \in \Lambda$.*

(b) *There exist distinct points ω_j, $|\omega_j|>1$, $j=1, 2, ..., m$, and $g \in A_0$ such that $L_{n-1}(\omega_j, g)=0$, $j=1, ..., m$, for every $n \in \Lambda$.*

The corollary follows because they can construct a function f in A_0 for which the highest degree term of $L_{n-1,1}(f, z)$ is zero for all n.

The close relationship between $s_{n-1}(f, z)$ and $L_{n-1,1}(f, z)$ led Ivanov and Saff in the remaining part of [4] to study results for $L_{n-1,1}(f, z)$ similar to a theorem of Jentzsch [5] (see also [7, 8]) that if f is in A_0, then every point z with $|z|=1$ is a limit point of zeros of $s_{n-1}(f, z)$. Writing $L_{n-1,1}(f, z)=\sum_{j=0}^{n-1} C(j, n)\, z^j$ and defining

$$\sigma(f, \theta) := \overline{\lim_{n\to\infty}} \max_{(1-\theta)\, n \leqslant j \leqslant n} |C(j, n)|^{1/n},$$

they used a theorem of Grothmann [3] to show that for any f in A_0, $\sigma(f, \frac{1}{3})=1$ and offered the conjecture that $\sigma(f, \theta)=1$ for any $0<\theta<1$. Based on the truth of this conjecture, they proved an analogue of Jentzsch's theorem for the zeros of $\{L_{n-1,1}(f, z)\}_{n=1}^\infty$.

In this paper we begin an extension of the above results to the Hermite interpolant $H_{r,n}(f, z)$ of degree $rn-1$ which interpolates the function f at the zeros of $(z^n-1)^r$. In Section 2 we study three differnt forms for expressing $H_{r,n}$. One of these forms is in terms of the fundamental polynomials for Hermite interpolation and in Section 3 we study these further, giving an explicit form for these fundamental polynomials in terms of Stirling numbers. Our main result is the following extension of Theorem A, which was proved for $r=2$ by Goodman and Sharma [2]. Here $\mathscr{A}_{r-1}$ denotes the class of functions $f(z)$ which are analytic in $|z|<1$ and $f^{(r-1)}(z)$ is continuous in $|z| \leqslant 1$.

THEOREM 1. *For any given positive integers m and r, there exist r homogeneous polynomials $P_0, P_1, ..., P_{r-1}$, each of degree $\frac{1}{2}(r-1)\, m(m-1)$ and symmetric in m variables, such that for any $n \geqslant mr$, any $f \in \mathscr{A}_{r-1}$, and for any m distinct nonzero points $\omega_1, ..., \omega_m$ such that $P_\nu(\omega_1, ..., \omega_m) \neq 0$ $(\nu=0, 1, ..., r-1)$, the following two statements are equivalent*:

(a) *The coefficients of the mr highest powers of z in the expansion of the Hermite interpolant* $H_{r,n}(f, z)$ *are zero.*

(b) *For every* $\nu = 0, 1, ..., r-1$ *the Hermite interpolant* $H_{r,n}(g_\nu, z)$ *of the function* $g_\nu(z) := f(z) \prod_{j=1}^{m} (z - \omega_j \eta^\nu)^r$ *vanishes at the m points* $\{\omega_j \eta^\nu\}_{j=1}^{m}$, *where* η *is a primitive* rth *root of unity.*

This result is proved in Section 5 and depends on some properties of $H_{r,n}$ which are derived in Section 4. So far we have been unable to apply this result as in [4]. If one could construct a function f in $\mathscr{A}_{r-1}$ for which the r highest degree terms in $H_{r,n}(f, z)$ are zero for all n, then Theorem 1 would show that for almost all $|z| > 1$, there are functions g_ν in $\mathscr{A}_{r-1}$ for which $\{H_{r,n}(g_\nu, z\eta^\nu)\}_{n=1}^{\infty}$ is identically zero for $\nu = 0, 1, ..., r-1$, where η is a primitive rth root of unity.

2. Explicit Forms of the Polynomials $H_{r,n}(f, z)$

Let $r, n \in \mathbb{N}$ be fixed and $\omega = e^{2\pi i/n}$. Let $f(z) \in \mathscr{A}_{r-1}$ and let $f(z) = \sum_{s=0}^{\infty} a_s z^s$. Denote by $H_{r,n}(f, z)$ the polynomial of degree $rn-1$ interpolating f at the zeros of $(z^n - 1)^r$, i.e.,

$$H_{r,n}^{(\rho)}(f, \omega^\nu) = f^{(\rho)}(\omega^\nu) \qquad \text{for} \quad \nu = 0, 1, ..., n-1; \quad \rho = 0, 1, ..., r-1. \tag{2.1}$$

Thus $H_{1,n}(f, z)$ is the Lagrange interpolant in the roots of unity.

One can write $H_{r,n}(f, z)$ explicitly in three different forms which we now discuss:

(a) *In terms of* $f^{(\rho)}(\omega^\nu)$ *and the fundamental polynomials,*

$$H_{r,n}(f, z) = \sum_{\rho=0}^{r-1} \sum_{\nu=0}^{n-1} f^{(\rho)}(\omega^\nu) \, \mathscr{L}_{\rho,\nu}(z), \tag{2.2}$$

where the fundamental polynomials $\mathscr{L}_{\rho,\nu}(z) \in \pi_{rn-1}$ are determined uniquely by the condition

$$\mathscr{L}_{\rho,\nu}^{(R)}(\omega^N) = \delta_{\rho,R} \delta_{\nu,N} \qquad \text{for} \quad R = 0, 1, ..., r-1; \quad N = 0, 1, ..., n-1.$$

Rotating the argument z with ω^ν $(\nu = 0, 1, ..., n-1)$ around the origin, one gets

$$\mathscr{L}_{\rho,\nu}(z) = \omega^{\rho\nu} \mathscr{L}_{\rho,0}(\omega^{-\nu} z), \qquad \nu = 0, 1, ..., n-1; \tag{2.3}$$

that is, one has to find only r polynomials $\mathscr{L}_\rho \equiv \mathscr{L}_{\rho,0}$ in (2.2). Thus (2.2) becomes

$$H_{r,n}(f, z) = \sum_{\rho=0}^{r-1} \sum_{\nu=0}^{n-1} f^{(\rho)}(\omega^\nu) \, \omega^{\rho\nu} \mathscr{L}_\rho(\omega^{-\nu} z). \tag{2.4}$$

When $r=1$,

$$\mathscr{L}_0(z)=l(z):=\frac{1}{n}\cdot\frac{z^n-1}{z-1}, \tag{2.5}$$

and when $r=2$,

$$\mathscr{L}_0(z)=l^2(z)\{1-(n-1)(z-1)\}, \qquad \mathscr{L}_1(z)=(z-1)\,l^2(z).$$

(b) *In terms of $f^{(\rho)}(\omega^\nu)$ and powers of z.* Expanding $\mathscr{L}_\rho(z)$ from (a) in powers of z, one gets

$$H_{r,n}(f,z)=\sum_{j=0}^{rn-1} A(j;r,n)\,z^j, \tag{2.6}$$

where the coefficients $A(j;r,n)$ depend only on j, r, n and the values of f and its derivatives at the roots of unity. When $r=1$,

$$A(j;1,n)=\frac{1}{n}\sum_{\nu=0}^{n-1}\omega^{-j\nu}f(\omega^\nu),$$

while when $r=2$,

$$A(j;2,n)=\begin{cases}\dfrac{n+j}{n^2}\displaystyle\sum_{\nu=0}^{n-1}\omega^{-j\nu}f(\omega^\nu)-\frac{1}{n^2}\sum_{\nu=0}^{n-1}\omega^{(-j+1)\nu}f'(\omega^\nu), & 0\leqslant j<n,\\[2ex] \dfrac{n-j}{n^2}\displaystyle\sum_{\nu=0}^{n-1}\omega^{-j\nu}f(\omega^\nu)+\frac{1}{n^2}\sum_{\nu=0}^{n-1}\omega^{(-j+1)\nu}f'(\omega^\nu), & n\leqslant j<2n.\end{cases}$$

(c) *In terms of (a_ν) and powers of z.* Expanding $f^{(\rho)}(z)$ in power series, one gets a new form for the coefficients $A(j;r,n)$ of (2.6), where $A(j;r,n)$ depends on a_s in a very simple manner. Denote by $p_{k,r}(z)$ the fundamental polynomials of Lagrange interpolation at points $0, 1, ..., r-1$; i.e.,

$$p_{k,r}(z)=\prod_{\substack{m=0\\ m\neq k}}^{r-1}\frac{z-m}{k-m}=\frac{p_r(z)}{(x-k)\,p_r'(k)}\in\pi_{r-1}, \tag{2.7}$$

where $p_r(z)=[z]_{r-1}$, $[z]_\rho:=z(z-1)\cdots(z-\rho+1)$. That is, we have

$$p_{k,r}(m)=\delta_{k,m}, \qquad k,m=0,1,...,r-1.$$

LEMMA 1. *For any $j=0,1,...,n-1$; $k=0,1,...,r-1$, we have*

$$A(j+kn;r,n)=\sum_{s=0}^{\infty} p_{k,r}(s)\,a_{j+sn}. \tag{2.8}$$

Proof. We first express $f^{(\rho)}(\omega^\nu)$ in terms of $\{a_m\}$. We have

$$f^{(\rho)}(z)=\sum_{m=0}^{\infty}[m]_\rho\, a_m z^{m-\rho}=\sum_{j=0}^{n-1}\sum_{s=0}^{\infty}[j+sn]_\rho\, a_{j+sn} z^{j+sn-\rho}$$

and hence

$$f^{(\rho)}(\omega^\nu)=\sum_{j=0}^{n-1}\omega^{\nu(j-\rho)}\sum_{s=0}^{\infty}[j+sn]_\rho\, a_{j+sn},$$

$$\rho=0,1,...,r-1;\quad \nu=0,1,...,n-1. \tag{2.9}$$

Now observe that

$$p_{k,r}(0)=p_{k,r}(1)=\cdots=p_{k,r}(k-1)=0. \tag{2.10}$$

Since $\{p_{k,r}(z)\}_{k=0}^{r-1}$ are the fundamental polynomials of Lagrange interpolation at 0, 1, ..., $r-1$ and since $[x]_\rho$ is a polynomial of degree $\rho\leqslant r-1$, we have

$$[j+sn]_\rho=\sum_{k=0}^{r-1}[j+kn]_\rho\, p_{k,r}(s) \tag{2.11}$$

for any $j, s, n\in\mathbb{N}$ and $\rho=0,1,...,r-1$.

Consider the polynomial $H(z)$ of degree $rn-1$,

$$H(z)=\sum_{j=0}^{n-1}\sum_{k=0}^{r-1}z^{j+kn}\sum_{s=0}^{\infty}p_{k,r}(s)\, a_{j+sn}.$$

Then for $\rho=0,1,...,r-1$; $\nu=0,1,...,n-1$,

$$\begin{aligned}H^{(\rho)}(\omega^\nu)&=\sum_{j=0}^{n-1}\omega^{\nu(j-\rho)}\sum_{k=0}^{r-1}[j+kn]_\rho\sum_{s=0}^{\infty}p_{k,r}(s)\, a_{j+sn}\\&=\sum_{j=0}^{n-1}\omega^{\nu(j-\rho)}\sum_{s=0}^{\infty}a_{j+sn}\sum_{k=0}^{r-1}[j+kn]_\rho\, p_{k,r}(s)\\&=\sum_{j=0}^{n-1}\omega^{\nu(j-\rho)}\sum_{s=0}^{\infty}a_{j+sn}[j+sn]_\rho\\&=f^{(\rho)}(\omega^\nu),\end{aligned} \tag{2.12}$$

where we have successively used (2.10), (2.11), and (2.9). Thus $H(z)=H_{r,n}(f,z)$ and (2.8) follows from (2.6).

In particular we see from Lemma 1 that when $r=1$,

$$A(j;1,n)=\sum_{s=0}^{\infty}a_{j+sn},\qquad j=0,1,...,n-1,$$

and when $r=2$, we have

$$A(j;2,n)=\begin{cases}\sum\limits_{s=0}^{\infty}(1-s)\,a_{j+sn}, & 0\leqslant j<n,\\ \sum\limits_{s=0}^{\infty}(1+s)\,a_{j+sn}, & n\leqslant j<2n.\end{cases}$$

3. The Explicit Form of $\mathscr{L}_{\rho,\nu}(z)$

We shall now find the explicit form of the fundamental polynomials $\mathscr{L}_{\rho,\nu}(z)$. It is known that the form of the polynomials $\mathscr{L}_{\rho}(z)$ is given by

$$\mathscr{L}_{\rho}(z)=\frac{(l(z))^{r}}{\rho!}\sum_{\nu=0}^{r-1-\rho}b_{\rho,\nu}(z-1)^{\rho+\nu}\qquad(\rho=0,1,...,r-1).\tag{3.1}$$

We first note that the coefficients $b_{\rho,\nu}$ are independent of ρ. Define a sequence of numbers $\{b_\nu\}_0^\infty$ by the recurrence relation

$$b_0=1,\qquad b_\nu=-\sum_{\mu=0}^{\nu-1}\frac{b_\mu}{(\nu-\mu)!}(l^r)^{(\nu-\mu)}(1),\qquad \nu=1,2,...,\tag{3.2}$$

where l is given by (2.5).

We shall now prove the following.

Lemma 2. *The coefficients* $\{b_{\rho,\nu}\}_{\rho=0}^{r-1}$ *in* (3.1) $(\nu=0,1,...,r-\rho-1)$ *are given by*

$$b_{\rho,\nu}=b_\nu,\tag{3.3}$$

where the sequence $\{b_\nu\}_0^\infty$ *is given by* (3.2).

Proof. From (2.3) we see that for any integer ρ, $0\leqslant\rho\leqslant r-1$, we have

$$1=\mathscr{L}_{\rho}^{(\rho)}(1)=b_{\rho,0}.$$

For $\nu=1,...,r-\rho-1$, we see by using Leibniz rule that

$$\begin{aligned}0=\mathscr{L}_{\rho}^{(\rho+\nu)}(1)&=\frac{1}{\rho!}\sum_{\mu=\rho}^{\rho+\nu}\binom{\rho+\nu}{\mu}(l^r)^{(\rho+\nu-\mu)}(1)\,\mu!\,b_{\rho,\mu-\rho}\\&=\frac{(\rho+\nu)!}{\rho!}\sum_{\mu=\rho}^{\rho+\nu}\frac{(l^r)^{(\rho+\nu-\mu)}(1)}{(\rho+\nu-\mu)!}b_{\rho,\mu-\rho}\\&=\frac{(\rho+\nu)!}{\rho!}\sum_{\mu=0}^{\nu}\frac{(l^r)^{(\nu-\mu)}(1)}{(\nu-\mu)!}b_{\rho,\mu};\end{aligned}$$

hence, we get

$$b_{\rho,\nu} = -\sum_{\mu=0}^{\nu-1} \frac{b_{\rho,\mu}}{(\nu-\mu)!} (l^r)^{(\nu-\mu)}(1), \qquad \nu = 1, ..., r-\rho-1.$$

Thus the coefficients $b_{\rho,\mu}$ satisfy relation (3.2) and this completes the proof. ∎

So from (3.1), (2.5), (3.3), and (2.3) we have

$$\mathscr{L}_{\rho,\nu}(z) = w^{\rho\nu} \frac{1}{\rho!\, n^r} \left(\frac{z^n - 1}{\omega^{-\nu} z - 1}\right)^r \sum_{j=0}^{r-1-\rho} b_j (\omega^{-\nu} z - 1)^{\rho+j} \tag{3.4}$$

for $\nu = 0, 1, ..., n-1$; $\rho = 0, 1, ..., r-1$.

We now give an explicit formula for the numbers b_j.

LEMMA 3. *Let $n, r \in \mathbb{N}$. For $j = 0, 1, ..., r-1$, the coefficient b_j in (3.4) is given by*

$$b_j = \frac{(r-j-1)!}{(r-1)!} \sum_{k=0}^{n} s_{r-k}^{(r)} t_{r-j}^{(r-k)} n^k,$$

where $s_i^{(j)}$, $t_i^{(j)}$ are the Stirling numbers of the first and second kind, respectively.

Proof. From Lemma 1 with $j = n-1$; $k = r-1$, we see that the coefficients of z^{nr-1} in $H_{r,n}(f, z)$ are

$$\sum_{\lambda=1}^{\infty} \binom{\lambda-1}{r-1} a_{\lambda n-1}. \tag{3.5}$$

From (2.4), (2.3), and (3.1) the same coefficient is

$$n^{-r} \sum_{\rho=0}^{r} \frac{1}{\rho!} b_{r-1,\rho} \sum_{\nu=0}^{n-1} \omega^{(\rho+1)\nu} f^{(\rho)}(\omega^\nu). \tag{3.6}$$

The Taylor expansion of f gives

$$\sum_{\nu=0}^{n-1} \omega^{(\rho+1)\nu} f^{(\rho)}(\omega^\nu) = n \sum_{\lambda=1}^{\infty} \binom{\lambda n-1}{\rho} \rho!\, a_{\lambda n-1}. \tag{3.7}$$

Replacing (3.7) in (3.6) and equating (3.6) to (3.5), we get

$$\sum_{\lambda=1}^{\infty} a_{\lambda n-1} n^{-r+1} \sum_{\rho=0}^{r-1} b_{r-1-\rho} \binom{\lambda n-1}{\rho} = \sum_{\lambda=1}^{\infty} \binom{\lambda-1}{r-1} a_{\lambda n-1}. \tag{3.8}$$

Because (3.8) is true for any $f \in C^{r-1}$, we obtain the system of equations for b_ρ's,

$$n^{-r+1} \sum_{\rho=0}^{r-1} b_{r-1-\rho} \binom{\lambda n - 1}{\rho} = \binom{\lambda - 1}{r-1}, \qquad \lambda = 1, 2, ...,$$

which after multiplication by λ can be rewritten as

$$\sum_{\rho=0}^{r-1} b_{r-1-\rho} n^{-r}(\rho+1) \binom{\lambda n}{\rho+1} = r \binom{\lambda}{r} \tag{3.9}$$

or

$$\sum_{\rho=1}^{r} b_{r-\rho} n^{-r} \rho \binom{\lambda n}{\rho} = r \binom{\lambda}{r}, \qquad \lambda = 1, 2, ...$$

In order to solve (3.9), we recall the definition and simple properties of Stirling numbers of the first and second kind (denoted here by $s_i^{(m)}$ and $t_i^{(m)}$ respectively):

$$\binom{x}{m} m! = \sum_{i=1}^{m} s_i^{(m)} x^i, \qquad x^m = \sum_{i=1}^{m} t_i^{(m)} \binom{x}{i} i! \tag{3.10}$$

Stirling numbers satisfy the following relations:

$$s_i^{(m)} = -(m-1)\, s_i^{(m)} + s_{i-1}^{(m-1)}, \qquad s_1^{(m)} = (-1)^{m-1}(m-1)! \tag{3.11}$$

$$s_m^{(m)} = 1;$$

$$t_i^{(m)} = i t_i^{(m-1)} + t_{i-1}^{(m-1)}, \qquad t_1^{(m)} = t_m^{(m)} = 1. \tag{3.12}$$

Using (3.10), we get

$$\begin{aligned} r\binom{\lambda}{r} &= \frac{1}{(r-1)!} \sum_{i=1}^{r} s_i^{(r)} \lambda! \\ &= \frac{1}{(r-1)!} \sum_{i=1}^{r} n^{-i} (n\lambda)^l\, s_i^{(r)} \\ &= \frac{1}{(r-1)!} \sum_{i=1}^{r} n^{-i} s_i^{(r)} \sum_{\rho=1}^{i} t_\rho^{(i)} \binom{n\lambda}{\rho} \rho! \\ &= \sum_{\rho=1}^{r} \binom{n\lambda}{\rho} \rho \frac{(\rho-1)!}{(r-1)!} \sum_{i=\rho}^{r} s_i^{(r)} t_\rho^{(i)} n^{-i}. \end{aligned}$$

Therefore,

$$b_{r-\rho} = \frac{(\rho-1)!}{(r-1)!} \sum_{i=\rho}^{r} s_i^{(r)} t_\rho^{(i)} n^{r-i}, \qquad \rho = 1, 2, ..., r,$$

and the proof is complete.

4. Some Properties of $H_{r,n}$

In this section we shall investigate properties of $H_{r,n}$ related to Theorem 1. First, in relation to (a) of this theorem we shall find formulae for the coefficients of the n highest powers of z in (2.6). We shall need the following

Lemma 4. *For $\rho = 0, 1, ..., r-1$, we have the representation*

$$n^r(z-1)^\rho\, l^r(z) = (-1)^\rho \sum_{\nu=0}^{nr-r+\rho} A_\nu^{(\rho)} z^\nu$$
$$= \sum_{\nu=0}^{nr-r+\rho} A_\nu^{(\rho)} z^{nr-r+\rho-\nu}, \tag{4.1}$$

where

$$A_\nu^{(\rho)} = (-1)^\rho A_{nr-r+\rho-\nu}^{(\rho)} \tag{4.2}$$

and, in particular,

$$A_\nu^{(\rho)} = \binom{r+\nu-\rho-1}{\nu}, \qquad \nu = 0, 1, ..., n-1. \tag{4.3}$$

Proof. Since $l(z) = (z^n - 1)/(z-1) \cdot (1/n) = z^{n-1} l(z^{-1})$, it follows that

$$n^r(z-1)^\rho\, l^r(z) = n^r z^\rho (-1)^\rho\, (z^{-1}-1)^\rho\, z^{nr-r} l^r(z^{-1})$$
$$= z^{nr-r+\rho} \sum_{\nu=0}^{nr-r+\rho} A_\nu^{(\rho)} (z^{-1})^\nu$$
$$= \sum_{\nu=0}^{nr-r+\rho} A_\nu^{(\rho)} z^{nr-r+\rho-\nu}.$$

Comparing the above with (4.1) gives (4.2).

From (4.1) for $|z| < 1$, we obtain

$$n^r(z-1)^\rho\, l^r(z) = (-1)^\rho\, (1-z^n)^r\, (1-z)^{-(r-\rho)}$$
$$= (-1)^\rho\, (1-z^n)^r \sum_{\mu=0}^{\infty} \binom{r-\rho+\mu-1}{\mu} z^\mu.$$

If we compare the coefficients of z^μ ($\mu = 0, 1, ..., n-1$) in the above and in (4.1), we obtain (4.3).

Remark. The values of $A_\nu^{(\rho)}$ for $n \leqslant \nu \leqslant nr - r + \rho - n$ can be determined from the above, if necessary.

LEMMA 5. *For positive integers r, n,*

$$H_{r,n}(f,z)=\sum_{j=0}^{rn-1} A(j)\,z^j, \tag{4.4}$$

where $A(j)\equiv A(j;r,n)$ *and for* $j=1,2,...,n$,

$$A(rn-j)=n^{-r}\sum_{\rho=0}^{r-1} X_{\rho,\rho+j}\sum_{v=0}^{r-\rho-1} b_v\binom{j-1}{\rho+v+j-r}, \tag{4.5}$$

$$X_{\rho,\rho+j}=\sum_{k=0}^{n-1}\frac{f^{(\rho)}(\omega^k)}{\rho!}\,\omega^{k(\rho+j)}. \tag{4.6}$$

Proof. From (2.2), (3.4), and Lemma 4 we have

$$\begin{aligned}
H_{r,n}(f,z)&=\sum_{k=0}^{n-1}\sum_{\rho=0}^{r-1} f^{(\rho)}(\omega^k)\,\mathscr{L}_{\rho,k}(z)\\
&=\sum_{k=0}^{n-1}\sum_{\rho=0}^{r-1}\frac{f^{(\rho)}(\omega^k)}{\rho!}\sum_{v=0}^{r-\rho-1} b_v(l(z\omega^{-k}))^r\,(z\omega^{-k}-1)^{\rho+v}\,\omega^{k\rho}\\
&=\sum_{k=0}^{n-1}\sum_{\rho=0}^{r-1}\frac{f^{(\rho)}(\omega^k)}{\rho!}\frac{1}{n^r}\sum_{v=0}^{r-\rho-1} b_v\omega^{k\rho}\\
&\quad\times\sum_{\mu=0}^{nr-r+\rho+v} A_\mu^{(\rho+v)}z^{nr-r+\rho+v-\mu}\omega^{-k(nr-r+\rho+v-\mu)}.
\end{aligned}$$

Putting $j=r+\mu-\rho-v$ in the two summations in v and μ, we obtain

$$\begin{aligned}
H_{r,n}(f,z)&=\sum_{k=0}^{n-1}\sum_{\rho=0}^{r-1}\frac{f^{(\rho)}(\omega^k)}{\rho!}\,n^{-r}\sum_{j=1}^{nr} z^{nr-j}\sum_{\mu=0}^{j-1} A_\mu^{(r+\mu-j)}\omega^{k(\rho+j)}b_{r+\mu-\rho-j}\\
&=\sum_{k=0}^{n-1}\sum_{\rho=0}^{r-1}\frac{f^{(\rho)}(\omega^k)}{\rho!}\,n^{-r}\sum_{j=1}^{nr} z^{nr-j}\omega^{k(\rho+j)}\sum_{l=r-\rho-j}^{r-\rho-1} b_l A_{\rho+l+j-r}^{(\rho+l)}
\end{aligned}$$

on putting $l=r+\mu-\rho-j$ and setting $b_l=0$ for $l<0$.

Recalling (4.6) we get

$$H_{r,n}(f,z)=n^{-r}\sum_{j=1}^{nr} z^{nr-j}\sum_{\rho=0}^{r-1} X_{\rho,\rho+j}\sum_{l=r-\rho-j}^{r-\rho-1} b_l A_{\rho+l+j-r}^{(\rho+l)}. \tag{4.7}$$

Then (4.7) and (4.3) give the result.

Now in relation to (b) of Theorem 1, we take any complex number τ and consider

$$g(z):=f(z)(z-\tau)^r\,q(z), \tag{4.8}$$

where $q(z)$ is a polynomial of degree $mr-r$ given by

$$q(z) := \sum_{\alpha=0}^{mr-r} \sigma_\alpha z^\alpha. \tag{4.9}$$

Also set $\sigma_\alpha = 0$ for $\alpha < 0$ or $\alpha > mr-r$.

THEOREM 2. *For positive integers r, n,*

$$H_{r,n}(g,\tau) = -(\tau^n-1)^r \sum_{j=1}^{rm} \sum_{\beta=0}^{r-1} (-1)^\beta \tau^\beta \sigma_{\beta+j-r} \binom{r-1}{\beta} A(rn-j). \tag{4.10}$$

For the proof of Theorem 2 we shall need the following.

LEMMA 6. *If r, p, β, j are nonnegative integers such that $0 \leqslant p \leqslant r-1$ and $r-j \leqslant \beta \leqslant r-1$, then the following identity holds:*

$$\sum_{\rho=0}^{r-p-1} \sum_{i=0}^{\rho} (-1)^\rho \binom{r}{\rho-i} \binom{\beta+j-r}{i} \binom{p+i}{\beta}$$
$$= (-1)^{p+r-1} \binom{r-1}{\beta} \binom{j-1}{r-p-1}. \tag{4.11}$$

Proof. Denoting the left side in (4.11) by $P(r, p, \beta, j)$ we see, after interchange of the order of summation, that

$$P(r,p,\beta,j) = \sum_{i=0}^{r-p-1} \binom{\beta+j-1}{i} \binom{p+i}{\beta} \sum_{\rho=1}^{r-p-1} (-1)^\rho \binom{r}{\rho-i}$$
$$= \sum_{i=0}^{r-p-1} \binom{\beta+j-r}{i} \binom{p+i}{\beta} (-1)^i \sum_{k=0}^{r-p-1-i} (-1)^k \binom{r}{k}.$$

Using the known identity

$$\sum_{k=0}^{m} (-1)^k \binom{r}{k} = (-1)^m \binom{r-1}{m}$$

(which can also be easily proved by indunction on m), we obtain

$$P(r,p,\beta,j) = \sum_{i=0}^{r-p-1} \binom{\beta+j-r}{i} \binom{p+i}{\beta} (-1)^{r-p-1} \binom{r-1}{r-p-1-i}$$
$$= (-1)^{r-p-1} \frac{(\beta+j-r)!\,(r-1)!}{(p+j-r)!\,(r-1-p)!\,\beta!}$$
$$\times \sum_{i=0}^{r-p-1} \binom{r-p-1}{i} \binom{p+j-r}{\beta+j-r-i}.$$

Since

$$\sum_{i=0}^{r-p-1} \binom{r-p-1}{i} \binom{p+j-r}{\beta+j-r-i}$$

is the coefficient of $x^{\beta+j-r}$ in the product $(1+x)^{r-p-1}(1+x)^{p+j-r}$, i.e., $(1+x)^{j-1}$, we see that

$$P(r, p, \beta, j) = (-1)^{r-p-1} \frac{(\beta+j-r)!\,(r-1)!}{(p+j-r)!\,(r-p-1)!\,\beta!} \binom{j-1}{\beta+j-r}$$

which reduces to the right side in (4.11). ∎

Proof of Theorem 2. By the Leibniz formula, we have

$$\begin{aligned} g^{(\rho)}(z) &= \sum_{l=0}^{\rho} \binom{\rho}{l} f^{(l)}(z)\, D_z^{\rho-l}((z-\tau)^r\, q(z)) \\ &= \sum_{l=0}^{\rho} \binom{\rho}{l} f^{(l)}(z) \sum_{i=0}^{\rho-l} \binom{\rho-l}{i} \frac{r!}{(r-i)!} (z-\tau)^{r-i}\, q^{(\rho-l-i)}(z). \end{aligned} \tag{4.12}$$

From (2.2) and (3.4) we get

$$H_{r,n}(g, \tau) = \left(\frac{\tau^n - 1}{n}\right)^r \sum_{\rho=0}^{r-1} \sum_{k=0}^{n-1} \frac{\omega^{kr}}{(\tau-\omega^k)^r} S(\rho, k), \tag{4.13}$$

where

$$S(\rho, k) := \sum_{v=0}^{k-1-\rho} b_v \frac{(\tau-\omega^k)^{\rho+v}}{\rho!} \omega^{-vk} g^{(\rho)}(\omega^k). \tag{4.14}$$

From (4.12), we get

$$g^{(\rho)}(\omega^k) = \sum_{l=0}^{\rho} \binom{\rho}{l} f^{(l)}(\omega^k) \sum_{i=0}^{\rho-l} \binom{\rho-l}{i} \frac{r!}{(r-i)!} (\omega^k-\tau)^{r-i}\, q^{(\rho-l-i)}(\omega^k). \tag{4.15}$$

Combining (4.13), (4.14), and (4.15), we obtain

$$H_{r,n}(g, \tau) = \left(\frac{\tau^n - 1}{n}\right)^r S_1,$$

where

$$S_1 = \sum_{\rho=0}^{r-1} \sum_{k=0}^{n-1} \frac{\omega^{kr}}{(\tau-\omega^k)^r} \sum_{v=0}^{r-1-\rho} b_v \frac{(\tau-\omega^k)^{\rho+v}}{\rho!} \omega^{-kv} \sum_{l=0}^{\rho} \binom{\rho}{l} f^{(l)}(\omega^k)$$
$$\times \sum_{i=0}^{\rho-l} \binom{\rho-l}{i} \frac{r!}{(r-i)!} (\omega^k-\tau)^{r-i} q^{(\rho-l-i)}(\omega^k).$$

Interchanging the order of summation in $\sum_{\rho=0}^{r-1}$ and $\sum_{l=0}^{\rho}$, we derive

$$S_1 = \sum_{l=0}^{r-1} \sum_{k=0}^{n-1} \omega^{rk} f^{(l)}(\omega^k) \sum_{\rho=l}^{r-1} \sum_{v=0}^{r-\rho-1} \sum_{i=0}^{\rho-l} F_v(l,k,\rho,i), \tag{4.16}$$

where we have set

$$F_v(l,k,\rho,i) := b_v \binom{\rho}{l}\binom{\rho-l}{i} \frac{r!}{(r-i)!} (-1)^{r-i} \frac{(\tau-\omega^k)^{\rho+v-i}}{\rho!}$$
$$\times q^{(\rho-l-i)}(\omega^k)\, \omega^{-kv}$$
$$= b_v(-1)^{r-i} (\tau-\omega^k)^{\rho+v-i} \frac{\omega^{-kv}}{l!(\rho-l-i)!} \binom{r}{i} q^{(\rho-l-i)}(\omega^k). \tag{4.17}$$

Putting $\rho+l$ for ρ in (4.16), we obtain

$$S_1 = \sum_{l=0}^{r-1} \sum_{k=0}^{n-1} \frac{f^{(l)}(\omega^k)}{l!} \sum_{\rho=0}^{r-l-1} \sum_{v=0}^{r-l-\rho-1} b_v$$
$$\times \sum_{i=0}^{\rho} \omega^{k(r-v)}(-1)^{r-i} (\tau-\omega^k)^{\rho+v-i+l} \binom{r}{i} \frac{q^{(\rho-i)}(\omega^k)}{(\rho-i)!}$$
$$= \sum_{l=0}^{r-1} \sum_{k=0}^{n-1} \frac{f^{(l)}(\omega^k)}{l!} \sum_{\rho=0}^{r-l-1} \sum_{v=0}^{r-l-\rho-1} b_v$$
$$\times \sum_{i=0}^{\rho} \omega^{k(r-v)}(-1)^{r-\rho+i} (\tau-\omega^k)^{v+i+l} \binom{r}{\rho-i} \frac{q^{(i)}(\omega^k)}{i!}. \tag{4.18}$$

Recalling the value of $q(z)$ from (4.9), we see that

$$I(z) := z^{r-v}(\tau-z)^{v+i+l} \frac{q^{(i)}(z)}{i!}$$
$$= z^{r-v} \sum_{\alpha=0}^{mr-r} \binom{\alpha}{i} \sigma_\alpha z^{\alpha-i} \sum_{\beta=0}^{l+v+i} \binom{l+v+i}{\beta} \tau^{l+v+i-\beta}(-1)^\beta z^\beta$$
$$= \sum_{\alpha=0}^{mr-r} \binom{\alpha}{i} \sigma_\alpha \sum_{\beta=0}^{l+v+i} \binom{l+v+i}{\beta} \tau^{l+v+i-\beta}(-1)^\beta z^{\beta+\alpha+r-v-i}.$$

Putting $\gamma=\beta+\alpha+r-v-i$ and interchanging the order of summation, we have

$$I(z)=\sum_{\gamma=0}^{mr+l} z^{\gamma} \sum_{\alpha=0}^{\gamma+i+v-r} \binom{\alpha}{i} \sigma_{\alpha} \binom{l+v+i}{l+\alpha+r-\gamma} (-1)^{\gamma+i+v-\alpha-r} \tau^{l+\alpha+r-\gamma}.$$

Using the above expression for $I(z)$ in (4.18) with z replaced by ω^k we get

$$S_1=\sum_{l=0}^{r-1} \sum_{k=0}^{n-1} \frac{f^{(l)}(\omega^k)}{l!} \sum_{\gamma=0}^{rm+l} \omega^{k\gamma} \sum_{\rho=0}^{r-l-1} \sum_{v=0}^{r-l-\rho-1} b_v$$
$$\times \sum_{i=0}^{\rho} \binom{r}{\rho-i} \sum_{\alpha=0}^{\gamma+i+v-r} \binom{\alpha}{i} \sigma_{\alpha} \binom{l+v+i}{l+\alpha+r-\gamma} (-1)^{\gamma+v-\alpha-\rho} \tau^{l-\gamma+\alpha+r}.$$

Since $i \leqslant \rho$ and $v \leqslant r-\rho-l-1$, it follows that $l+v+i \leqslant l+v+\rho \leqslant r-1<r$, so that for $\gamma \leqslant l$, we have $l-\gamma+\alpha+r \geqslant r>l+v+i$. Thus we have $\binom{l+v+i}{l-\gamma+\alpha-r}=0$ for $\gamma \leqslant l$.

Recalling (4.6), we see on setting $\beta:=l-\gamma+\alpha+r$ that

$$S_1=\sum_{l=0}^{r-1} \sum_{\gamma=l+1}^{rm+l} X_{l,\gamma} \sum_{\rho=0}^{r-l-1} \sum_{v=0}^{r-\rho-l-1} b_v \sum_{i=0}^{\rho} \binom{r}{\rho-i} T(i, v, \rho, l, \gamma), \tag{4.19}$$

where we have set

$$T(i, v, \rho, l, \gamma):=\sum_{\beta=0}^{l+i+v} \binom{\beta+\gamma-l-r}{i} \binom{l+v+i}{\beta} \sigma_{\beta+\gamma-l-r} \tau^{\beta} (-1)^{v-\rho+l+r-\beta}.$$

Putting $j=\gamma-l$, we have

$$T(i, v, \rho, l, j)=\sum_{\beta=0}^{l+i+v} \binom{\beta+j-r}{i} \binom{l+v+i}{\beta} \sigma_{\beta+j-r} \tau^{\beta} (-1)^{v-\rho+l+r-\beta}.$$

We shall change the order of summation successively in the expression for S_1 in (4.19). Thus

$$\sum_{i=0}^{\rho} \sum_{\beta=0}^{l+i+v}=\sum_{\beta=0}^{l+\rho+v} \sum_{i=0}^{\rho} \binom{r}{\rho-i} \binom{\beta+j-r}{i} \binom{l+v+i}{\beta} \sigma_{\beta+j-r} \tau^{\beta}.$$

Then we see that

$$\sum_{v=0}^{r-\rho-l-1} \sum_{\beta=0}^{l+\rho+v}=\sum_{\beta=0}^{r-1} \sum_{v=\beta-l-\rho}^{r-l-\rho-1}$$
$$=\sum_{\beta=0}^{r-1} \sum_{v=0}^{r-l-\rho-1} \binom{r}{\rho-i} \binom{\beta+j-r}{i} \binom{l+v+i}{\beta} \sigma_{\beta+j-r} \tau^{\beta}$$

because $\binom{l+v+i}{\beta}$ vanishes for $v<\beta-l-\rho$. Also,

$$\sum_{\rho=0}^{r-l-1}\sum_{v=0}^{r-l-\rho-1}=\sum_{v=0}^{r-l-1}\sum_{\rho=0}^{r-l-v-1}.$$

Combining all the above changes of order of summations, we finally arrive at the following value for S_1 after replacing γ by $j+l$ and after interchanging the first two summations in (4.19). Thus,

$$S_1=(-1)^r\sum_{j=1}^{rm}\sum_{l=0}^{r-1}(-1)^l X_{l,\,l+j}\sum_{\beta=0}^{r-1}(-1)^\beta\,\tau^\beta\sigma_{\beta+j-r}$$
$$\times\sum_{v=0}^{r-l-1}(-1)^v\,b_v S_2(v+l,\beta), \tag{4.20}$$

where

$$S_2(p,\beta):=\sum_{\rho=0}^{r-p-1}\sum_{i=0}^{\rho}(-1)^\rho\binom{r}{\rho-i}\binom{\beta+j-r}{i}\binom{p+i}{\beta},$$
$$=(-1)^{p+r-1}\binom{r-1}{\beta}\binom{j-1}{r-p-1}, \tag{4.21}$$

by Lemma 6.

To sum up we have shown that

$$H_{r,n}(g,\tau)=\left(\frac{\tau^n-1}{n}\right)^r S_1, \tag{4.22}$$

where by (4.20) and (4.21),

$$S_1=-\sum_{j=1}^{rm}\sum_{l=0}^{r-1}X_{l,\,l+j}\sum_{\beta=0}^{r-1}(-1)^\beta\,\tau^\beta\sigma_{\beta+j-r}\sum_{v=0}^{r-l-1}b_v\binom{r-1}{\beta}\binom{j-1}{r-l-v-1}$$
$$=-\sum_{j=1}^{rm}\sum_{\beta=0}^{r-1}(-1)^\beta\,\tau^\beta\sigma_{\beta+j-r}\binom{r-1}{\beta}\sum_{l=0}^{r-1}X_{l,\,l+j}\sum_{v=0}^{r-l-1}b_v\binom{j-1}{r-l-v-1}$$
$$=-\sum_{j=1}^{rm}\sum_{\beta=0}^{r-1}(-1)^\beta\,\tau^\beta\sigma_{\beta+j-r}\binom{r-1}{\beta}n^r A(rn-j),$$

by (4.5). Combining this with (4.22) then gives (4.10) and completes the proof.

5. Proof of Theorem 1

From (4.10) we see immediately that (a) implies (b). It remains to prove that (b) implies (a). Condition (b) asserts that if η is a primitive rth root of unity then the r functions $g_v(z)$, $v = 0, 1, ..., r-1$, given by

$$g_v(z) := f(z)(z-\omega_i\eta^v)^r \, q_{v,i}(z),$$
$$q_{v,i}(z) := \prod_{\substack{k=1\\ k\neq i}}^{m} (z-\omega_k\eta^v)^r \tag{5.1}$$

have the property that $H_{r,n}(g_v, z)$ vanish for $z = \omega_i\eta^v$ $(i = 1, 2, ..., m)$. Now g_v has the form (4.8) with $\tau = \omega_i\eta^v$, where $q(z)$ in (4.9) is replaced by

$$q_{v,i}(z) = \sum_{\alpha=0}^{mr-r} \sigma_{\alpha,i} z^\alpha \eta^{-v\alpha}, \tag{5.2}$$

where $\sigma_{\alpha,i}$ are symmetric functions in the variables $\{\omega_1, ..., \omega_m\}\backslash\{\omega_i\}$.

By (4.10), we have

$$\begin{aligned} H_{r,n}(g_v, \omega_i\eta^v) &= -(\omega_i^n\eta^{nv}-1)^r \sum_{j=1}^{rm}\sum_{\beta=0}^{r-1} (-1)^\beta \, \omega_i^\beta \eta^{v\beta} \sigma_{\beta+j-r,i} \binom{r-1}{\beta} \\ &\quad\times A(nr-j)\,\eta^{-v(\beta+j-r)} \\ &= -(\omega_i^n\eta^{nv}-1)^r \sum_{j=1}^{rm} \eta^{-vj} A(nr-j) \sum_{\beta=0}^{r-1} (-1)^\beta \, \omega_i^\beta \sigma_{\beta+j-r,i}\binom{r-1}{\beta}. \end{aligned} \tag{5.3}$$

Now

$$\begin{aligned} \prod_{l=1}^{m} (z-\omega_l)^r/(z-\omega_i) &= (z-\omega_i)^{r-1} \prod_{\substack{l=1\\ l\neq i}}^{m} (z-\omega_l)^r \\ &= \sum_{\alpha=0}^{rm-r} \sigma_{\alpha,i} z^\alpha \sum_{\beta=0}^{r-1} (-1)^\beta \, \omega_i^\beta \binom{r-1}{\beta} z^{r-1-\beta} \\ &= \sum_{j=0}^{rm-1} z^j \sum_{\beta=0}^{r-1} (-1)^\beta \, \omega_i^\beta \binom{r-1}{\beta} \sigma_{j-r+1+\beta,i}. \end{aligned} \tag{5.4}$$

Comparing (5.3) and (5.4), we see that

$$H_{r,n}(g_v, \omega_i\eta^v) = -\eta^{-v}(\omega_i^n\eta^{nv}-1)^r \sum_{j=0}^{rm-1} \eta^{-vj} A(rn-j-1)\, c_{j,i},$$

where $c_{j,i}$ are given by the generating function

$$\frac{\prod_{l=1}^{m}(z-\omega_l)^r}{z-\omega_i}=\sum_{j=0}^{rm-1} c_{j,i}z^j. \tag{5.5}$$

Thus to show that (b)⇒(a) is equivalent to showing that the system of equations

$$\sum_{j=0}^{rm-1} \eta^{-vj}c_{j,i}A(rn-j-1)=0, \qquad i=1, ..., m; \quad v=0, 1, ..., r-1, \tag{5.6}$$

is nonsingular.

If we multiply the system (6.7) by η^{vk} and sum with respect to v from 0 to $r-1$, we obtain r homogeneous systems of equations of m variables:

$$\sum_{\lambda=0}^{m-1} c_{\lambda r+j,i}A(nr-\lambda r-j-1)=0, \qquad i=1, ..., m, \tag{5.7}$$

for every $j=0, 1, ..., r-1$.

Let us denote the determinant of the system (5.7) by $\Delta_{j,m}:=\Delta_{j,m}(\omega_1, ..., \omega_m)=\det(c_{\lambda r+j,i})_{i=1,}^{m}{}_{\lambda=0}^{m-1}$. The coefficients $c_{j,i}$ are homogeneous polynomials in $\omega_1, ..., \omega_m$ of degree $rm-1-j$. In particular,

$$c_{rm-1,i}=1 \quad \forall i, \qquad c_{0,i}=(-1)^{rm}\frac{(\omega_1\cdots\omega_m)^r}{-\omega_i}, \quad \forall i$$

and

$$c_{j-1,i}-\omega_i c_{j,i}=(-1)^{rm-j}S_{rm-j},$$

where

$$\prod_{l=1}^{m}(z-\omega_l)^r=\sum_{j=0}^{rm} S_{rm-j}z^j.$$

So $\Delta_{j,m}(\omega_1, ..., \omega_m)$ is a polynomial in $\omega_1, ..., \omega_m$, homogeneous of degree

$$\sum_{\lambda=0}^{m-1}(rm-1-\lambda r-j)=\frac{r}{2}m(m+1)-m(j+1). \tag{5.8}$$

In order to prove that the system (5.7) is nonsingular, we shall need the following.

LEMMA 7. *For $m=1, 2, ...$ and for $j=0, 1, ..., r-1$, we have*

$$\Delta_{j,m}(\omega_1, ..., \omega_m)=(\omega_1\cdots\omega_m)^{r-1-j}\prod_{r<s}(\omega_r-\omega_s)\,P_{j,m}(\omega_1, ..., \omega_m), \tag{5.9}$$

where $P_{j,m}$ *is a homogeneous polynomial of degree* $\frac{1}{2}(r-1)\,m(m-1)$ *in* $\omega_1, ..., \omega_m$ *such that*

$$P_{j,1}(\omega_1) = (-1)^{r-1-j} \binom{r-1}{j}$$

and for $m = 2, 3, ...,$

$$P_{j,m}(\omega_1, ..., \omega_{m-1}, 0) = (-1)^{rm+m-j} (\omega_1 \cdots \omega_{m-1})^{r+1} \binom{r-1}{j} \times P_{j,m-1}(\omega_1, ..., \omega_{m-1}). \tag{5.10}$$

Proof. It follows from (5.5) that (5.9) holds for $m=1$. If we differentiable both sides of (5.5) k times ($k \leqslant r-1$) with respect to ω_i and then put $\omega_i = 0$, we get

$$\sum_{j=0}^{rm-1} z^j [D^k_{\omega_i} c_{j,i}]_{\omega_i=0} = \frac{(-1)^k (r-1)!}{(r-1-k)!} z^{r-1-k} \prod_{\substack{l=1\\ l\neq i}}^{m} (z-\omega_l)^r.$$

Comparing powers of z on both sides, we see that

$$[D^k_{\omega_i} c_{j,i}]_{\omega_i=0} = \begin{cases} 0, & j = 0, 1, ..., r-2-k, \\ \dfrac{(-1)^k (r-1)!}{(r-1-k)!} \displaystyle\prod_{\substack{l=1\\ l\neq i}}^{m} (-\omega_l)^r, & j = r-1-k. \end{cases}$$

Similarly, if we differentiate (5.5) with respect to ω_l ($l \neq i$) and then put $\omega_l = 0$, we see exactly as above that

$$[D^k_{\omega_l} c_{j,i}]_{\omega_l=0} = 0, \qquad j = 0, 1, ..., r-1-k. \tag{5.12}$$

So $c_{j,i}$ is divisible by $(\prod_{l=1}^m \omega_l)^{r-1-j}$ for every $i = 1, ..., m$. Thus every element of the first column of $\Delta_{j,m}(\omega_1, ..., \omega_m)$ is divisible by $(\prod_{l=1}^m \omega_l)^{r-1-j}$. If $\omega_r = \omega_s$ for some $r \neq s$, then Eq. (5.7) is identical for $i = r$ and s and so $\Delta_{j,m}$ vanishes. Thus $\Delta_{j,m}$ is divisible by $\prod_{r<s} (\omega_r - \omega_s)$ and this proves (5.9).

For the degree of $P_{j,m}$, we see from (5.9) and (5.8) that

$$\deg(P_{j,m}) = \deg \Delta_{j,m} - m(r-1-j) - \tfrac{1}{2} m(m-1) = \tfrac{1}{2}(r-1)\, m(m-1).$$

From (5.11) and (5.12) with $i = m$, $k = r-1-j$, we get

$$[D^{r-1-j}_{\omega_m} \Delta_{j,m}(\omega_1, ..., \omega_m)]_{\omega_m=0} = (-1)^{r-1-j} \frac{(r-1)!}{j!} \prod_{l=1}^{m-1} (-\omega_l)^r \Delta_{m,1}, \tag{5.13}$$

where $\Delta_{m,1}$ is the co-factor of the last element of the first column of $[\Delta_{j,m}]_{\omega_m=0}$, i.e.,

$$\begin{aligned}\Delta_{m,1} &= (-1)^{m+1} \det(c_{r\lambda+j,i} \mid_{\omega_m=0})_{i=1,\lambda=1}^{m-1\ m-1} \\ &= (-1)^{m+1} \Delta_{j,m-1}(\omega_1, ..., \omega_{m-1}). \qquad (5.14)\end{aligned}$$

We obtain the last equality from (5.5) as follows ($i \neq m$):

$$\begin{aligned}\frac{1}{z-\omega_i} \prod_{l=1}^{m-1} (z-\omega_l)^r &= z^{-r} \frac{1}{z-\omega_i} \prod_{l=1}^{m} (z-\omega_l)^r \mid_{\omega_m=0} \\ &= \sum_{j=0}^{rm-1} c_{j,i} \mid_{\omega_m=0} z^{j-r} \\ &= \sum_{j=0}^{r(m-1)-1} c_{r+j,i} \mid_{\omega_m=0} z^j.\end{aligned}$$

On the other hand, differentiating (5.9) $r-1-j$ times gives

$$\begin{aligned}D_{\omega_m}^{r-1-j} \Delta_{j,m} \mid_{\omega_m=0} = (r-1-j)! \left(\prod_{l=1}^{m-1} \omega_l \right)^{r-j} \prod_{\substack{r<s \\ r \neq m}} (\omega_r - \omega_s) \\ \times P_{j,m}(\omega_1, ..., \omega_{m-1}, 0). \qquad (5.15)\end{aligned}$$

Finally, comparing (5.13) and (5.14) with (5.15) gives (5.10). ▮

COROLLARY 1. *For* $m = 1, 2, 3, ...$ *and* $j = 0, 1, ..., r-1$ *the polynomials* $\Delta_{j,m}$ *and* $P_{j,m}$ *do not vanish identically.*

Proof. The result follows by induction on m by utilizing Lemma 7. ▮

In order to finish the proof of the theorem, we identify the polynomials P_j from the condition of the theorem by $P_{j,m}$. Then Corollary 1 says that condition

$$P_j(\omega_1, ..., \omega_m) \neq 0, \qquad j = 0, 1, ..., r-1,$$

is fulfilled for almost all $(\omega_1, ..., \omega_m)$ and then (5.9) asserts that system (5.7) is nonsingular for our $\omega_1, ..., \omega_m$. ▮

Finally we give the polynomials $P_{j,m}$ for simple cases. When $r = 1$ we see from Lemma 7 that $P_{j,m} = (-1)^{-jm}$. For $r = 2$, it was shown in [2] that $\Delta_{j,m}$, and hence $P_{j,m}$, is divisible by $\prod_{r<s} (\omega_r + \omega_s)$. Then Lemma 7 shows that

$$P_{j,m}(\omega_1, ..., \omega_m) = (-1)^{jm + (1/2)m(m+1)} \prod_{r<s} (\omega_r + \omega_s).$$

For $r \geqslant 2$, we have not derived any general formula for $P_{j,m}$. For $r=3$, $m=2$, a direct calculation shows that

$$P_{0,2}(\omega_1, \omega_2) = P_{2,2}(\omega_1, \omega_2) = \omega_1^2 + 4\omega_1\omega_2 + \omega_2^2,$$

$$P_{1,2}(\omega_1, \omega_2) = 4\omega_1^2 + 7\omega_1\omega_2 + 4\omega_2^2.$$

References

1. A. S. Cavaretta, A. Sharma, and R. S. Varga, Interpolation in the roots of unity: An extension of a theorem of Walsh, *Resultate Math.* **3** (1981), 155–191.
2. T. N. T. Goodman and A. Sharma, A property of Hermite interpolants in the roots of unity, *Ganita* **43** (1992), 171–180.
3. R. Grothmann, On the zeros of sequences of polynomials, *J. Approx. Theory* **6** (1990), 351–359.
4. K. G. Ivanov and E. B. Saff, Behavior of the Lagrange interpolants in the roots of unity, *in* "Computational Methods and Function Theory, Proceedings, Valparasio, 1989" (St. Ruscheweyh *et al.*, Eds.), pp. 81–87, Springer-Verlag, Berlin, 1990.
5. R. Jentzsch, "Untersuchungen zur Theorie analytischer Funktionen, "Inaugural-dissertation, Berlin, 1914; *Acta Math.* **4** (1917), 219–270.
6. C. A. Micchelli and T. Rivlin, An optimal recovery problem of Walsh's equiconvergence theory, *J. Complexity* **3** (1987), 312–330.
7. G. Szegö, Über die Nullstellen van Polynomen, die in einem Kreis gleichmässig konvergieren, *Sitzungsber. Berlin Math. Ges.* **21** (1922), 53–64.
8. E. C. Titchmarsh, "The Theory of Functions," 2nd ed., Oxford Univ. Press, Oxford, 1939.
9. J. L. Walsh, "Interpolation and Approximation by Rational Functions in the Complex Plane," Amer. Math. Soc. Colloq. Publ. 20, 5th ed., Amer. Math. Soc., Providence, RI, 1969.